

# Spin flip of multiqubit states in discrete phase space

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Time reversal and spin flip are discrete symmetry operations of substantial import to quantum information and quantum computation. Spin flip arises in the context of separability, quantification of entanglement and the construction of Universal NOT gates. The present work investigates the relationship between the quantum state of a multiqubit system represented by the Discrete Wigner Function (DWFs) and its spin-flipped counterpart. The two are shown to be related through a Hadamard matrix that is independent of the choice of the quantum net used for the tomographic reconstruction of the DWF. These results would be of interest to cases involving the direct tomographic reconstruction of the DWF from experimental data and in the analysis of entanglement related properties purely in terms of the Discrete Wigner function.

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## I. INTRODUCTION

Spin flip and time reversal are involution symmetry operations which frequently arise in quantum information studies. These are anti-unitary operations which are physically unrealizable in an idealized sense [1]. However such operations are critically important for entanglement detection, quantum optics, quantum computation and in the definition of certain entanglement measures [2–8]. Phase space representation of quantum states through Wigner functions provides a natural setting for understanding time reversal. However, in the case of qubit systems the relationship between the Discrete Wigner function and its spin-flipped/time reversed counterpart is ill understood, and the present work is an attempt at filling this gap. To appreciate how spin flip arises in different contexts, we shall consider the examples cited earlier in some detail. It is generally a difficult problem to establish whether a mixed state is entangled or separable. However, the celebrated Peres-Horodecki criterion, based on positivity of partial transposition (PPT) [2, 3], provides the necessary and sufficient condition for the separability of bipartite systems of dimension  $2 \otimes 2$ ,  $2 \otimes 3$  and Gaussian states [4]. Partial transposition for a  $2 \otimes 2$  system is defined by  $\rho_{ij,kl}^T = \rho_{il,kj}$ , where  $\rho$  is the density matrix of the system. Partial transposition essentially amounts to a spin flip/reflection of the second particle through a plane in the Poincaré sphere [1]. On a single qubit, this operation is represented by the operator  $\sigma_y \mathcal{C}$ , where  $\mathcal{C}$  is the complex conjugation of the state in the computational basis and  $\sigma_y$ , the usual Pauli operator. In the language of quantum maps,  $\rho$  is separable if and only if selective spin flip on the subsystem is a positive map. For a single qubit system represented by the pure state vector  $|\psi\rangle$ , the spin flipped state is defined by  $|\tilde{\psi}\rangle = (-i\sigma_y)|\psi^*\rangle$  and likewise, for the density matrix, it is defined as  $\tilde{\rho} = \sigma_y \rho^* \sigma_y$ . Unitary evolution constitutes a completely positive map which takes quantum states to quantum states and in hindsight, it is not surprising that only anti-unitary operations are effective for entanglement detection. In the context of quantum optics, the reflection of an elliptically polarized photon at a mirror, resulting in a polarization state orthogonal to it, is analogous to a spin-flip. This operation transforms the polarization state of the photon to the anti-podal one in the Poincaré sphere. In quantum computation and quantum simulation, the construction of Universal NOT (U-NOT) gates for selectively flipping a single qubit, though not perfectly feasible, is essential. Finally, since entanglement is viewed as a resource, there is a strong requirement to quantify it. Several entanglement measures such as negativity, concurrence, tangles and their generalizations to qudit and multiqubit systems have been proposed in the literature [5, 9–13]. The definition of these measures critically hinge on the spin flip operation. For example, for arbitrary bipartite mixed states, Wootters [5] has derived an entangled measure called concurrence, given by the expression:

$$C(\rho) = \max \left\{ 0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \right\} \quad (1)$$

where the  $\lambda$ 's are the eigenvalues of  $R = \rho \tilde{\rho}$  and  $\tilde{\rho}$  is the spin flipped state defined as:

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y) \quad (2)$$

Concurrence is an entanglement monotone viz., a quantity which is invariant under local quantum operations and classical communication. Given the usefulness of this quantity for entanglement quantification, there have been attempts at generalizing its definition for systems of higher dimensions and for multipartite states [6, 14]. For instance, the n-qubit concurrence  $C_n(\rho)$  is defined in terms of the eigenvalues of the  $R = \rho \tilde{\rho}$ , where n-qubit spin flip operation is defined as:

$$\tilde{\rho} = \sigma_y^{\otimes n} \rho^* \sigma_y^{\otimes n} \quad (3)$$

Though attempts at investigating entanglement have largely been based on representing the state through the density matrix, it by no means the unique way of doing so. In fact, alternate representations of the state through quasi-probability distributions such as the Wigner functions and through Stokes vectors are prevalent in quantum optics. Both these quantities can be readily reconstructed from measured data. These quantities are therefore valid representations of the state in their own right. Wigner functions have been used to investigate quantum states of light such as entangled, squeezed and photon-added coherent states [15, 16]. These are normalized and real valued functions which can however assume negative values over restricted regions of the domain, and are therefore called quasi-probability distributions [17, 18]. Indeed, the negativity of the Wigner function is taken to be a signature of the non-classicality of the state and is indicative of quantum interference effects [19]. Besides providing a visual presentation of these effects, Wigner functions have been used for investigating quantum dynamics, quantum random walks, decoherence and entanglement detection of continuous systems [4, 20]. The representation of the state through Wigner functions has the advantage that time reversal has a very transparent interpretation. The extension of the Peres-Horodecki criterion to bipartite Gaussian states by Simon [4] is based on the critical observation that transposition in the continuous case, is geometrically interpreted as a mirror reflection in the phase space. This is evident from the

observation that transformation  $\rho \rightarrow \rho^T$ , corresponds to the associated Wigner function transforming as:  $W(q, p) \rightarrow W(q, -p)$ . Since quantum information and computation applications generally use qubits, Discrete Wigner functions (DWFs) have been investigated in the literature. Unlike continuous Wigner functions, Discrete Wigner functions are not unique, and since the underlying field is discrete, certain restrictions are imposed. Constructions based on discrete  $2d \times 2d$  [21, 22] and  $d \times d$  grids [23–27] have been proposed in the literature. The DWF construction for the odd dimensional systems are given by Gross [28] and Chaturvedi et al [29]. Unlike the continuous case, there is no clear link between negativity and non-classicality in the case of DWFs. Even for maximally entangled bipartite states, the DWF is not necessarily negative for every choice of the quantum net. Despite this limitation, DWFs have been used to investigate stabilizer states which arise in the context of error correcting codes [30, 31], quantum optics [32], quantum state tomography [33, 34] and teleportation [35]. The construction proposed by Wootters [36–38], Gibbons et al., [39] for  $d \times d$  is particularly well suited for these studies and we shall use the same in the present work. In investigations involving the representation in terms of the DWF  $W$ , the spin flipped DWF  $\tilde{W}$  is required to qualitatively and quantitatively evaluate the state. Given this background, there is strong motivation to examine the relationship between  $W$  and  $\tilde{W}$  and to provide a prescription for computation of the latter from the former. Towards this end, in the present work, we exhibit an elegant relationship between  $W$  and  $\tilde{W}$ . We show that the  $n$  qubit DWF is related to its spin flipped counterpart through an  $N^2 \times N^2$  Hadamard matrix, where  $N = 2^n$ . Since the construction of the DWF depends on the Mutually Unbiased Basis sets (MUBS) that are employed, the DWF that is associated with a given density matrix is no longer unique as in the continuous case. Different bases sets lead to different choices of the so-called quantum net. It is therefore essential to show that the construction is independent of the quantum net and the proof for a general  $n$  - qubit system is also provided. The rest of the paper is structured as follows: Section II provides a quick overview of the Discrete Wigner function and Wootters' construction. The spin flipped DWF is derived in two steps in Section III. In section A, we show that  $W^{(*)}$ , the DWF of  $\rho^*$  is related to  $W$  through a Hadamard matrix and III-B and we show that this is independent of the choice quantum net. In III-C the DWF of the spin flipped density matrix is obtained by effecting a generalized shift in  $W^{(*)}$ . The final transformation matrix relating  $\tilde{W}$  to  $W$  also turns out to be Hadamard matrix which, is once again independent of the quantum net. Section IV-A illustrates the method for a single qubit system. Section IV-B outline the derivation for the two qubit case. The final section provides the conclusions with some brief remarks.

## II. DISCRETE WIGNER FUNCTION

This construction applies to Hilbert space of dimension  $N$  where,  $N$  is prime number or  $N = r^n$ , where  $r$  is prime. The DWF is then defined over an  $N \times N$  array of discrete points defined over a Galois field  $\mathcal{F}_N$ . The points are labelled by an ordered set  $\alpha = (p, q)$ . Since our interest is primarily in qubit states, the dimension of the Hilbert's space is  $N = 2^n$ . This construction uses the notion of lines, translational covariance and marginals in analogy with Wigner functions of continuous systems. A line is defined by as collection of points solving the equation  $aq + bp = c$ . A set of lines with identical  $a$  and  $b$  but with a different  $c$  are said to be parallel to each other. In Euclidean geometry, parallel lines do not share any point and non-parallel lines intersect at exactly one point. To ensure this, the underlying finite field structure has to be correctly chosen [39]. In this construction, there are  $N + 1$  sets of parallel lines with exactly  $N$  lines in each of them. Each set of parallel lines is called a striation. Crucially, a rank one projector  $P_\lambda = |\lambda\rangle \langle \lambda|$  is associated with each line  $\lambda$ , and the sum of the the Wigner function over all points  $\alpha$  contained in the line is required to satisfy the condition:

$$\sum_{\alpha \in \lambda} W_\alpha = \text{Tr}(\rho P_\lambda) \quad (4)$$

where  $\rho$  is quantum state. Self adjoint operators  $\hat{A}_\alpha$  with unit trace, having the property:  $\text{Tr}(\hat{A}_\alpha) = 1$  and  $\text{Tr}(\hat{A}_\alpha \hat{A}_\beta) = N\delta_{\alpha\beta}$ , are associated with each point of the lattice. With this association, the density operator may be written as:

$$\hat{\rho} = \sum_{\alpha} W_\alpha \hat{A}_\alpha \quad (5)$$

Thus, the  $\hat{A}_\alpha$  operators constitute an orthonormal basis and Wigner function are expansion coefficients. The projection operators are related to the operators  $\hat{A}_\alpha$  as:

$$P_\lambda = \frac{1}{N} \sum_{\alpha \in \lambda} \hat{A}_\alpha \quad (6)$$

This set of orthonormal vectors  $\{|\lambda_j^\kappa\rangle\}$  associated with the striation  $\kappa$  constitutes a basis set. There are therefore  $N + 1$  basis sets  $B^\kappa$ :  $\{\lambda_1^\kappa, \lambda_2^\kappa, \dots, \lambda_N^\kappa\}$  with the property:

$$\left| \langle \lambda_j^\kappa | \lambda_{j'}^{\kappa'} \rangle \right|^2 = \frac{1}{N} \left( 1 - \delta_{\kappa, \kappa'} \right) + \delta_{\kappa, \kappa'} \delta_{j, j'} \quad (7)$$

Such a set therefore constitutes a mutually unbiased basis set (MUBS) [40][36]. The expression for the Wigner function  $W_\alpha$  at every point  $\alpha$  can now be readily supplied:

$$W_\alpha = \frac{1}{N} \text{Tr} \left( \hat{\rho} \hat{A}_\alpha \right) \quad (8)$$

where:

$$\hat{A}_\alpha = \frac{1}{N} \left( \sum_{\lambda_j^\kappa \ni \alpha} \hat{P}_j^\kappa - I \right) \quad (9)$$

The important point is that the outcome probabilities associated with projective measurements along a line are:

$$p_j^\kappa = \text{Tr} \left( \hat{\rho} \hat{P}_j^\kappa \right) \quad (10)$$

and the Wigner function itself can be tomographically constructed through repeated measurements along all the lines as:

$$W_\alpha = \frac{1}{N} \left( \sum_{\lambda_j^\kappa \ni \alpha} p_j^\kappa - I \right) \quad (11)$$

To ensure translational covariance of the states, we associate a unitary operator  $\hat{T}_{(x,y)}$  with every phase space point with the condition that the composition law  $\hat{T}_\alpha \hat{T}_\beta = T_{\alpha+\beta}$  applies. The general unitary translation operator in phase space can be written as,

$$T_{(q,p)} = \sigma_x^{q_1} \sigma_z^{p_1} \otimes \sigma_x^{q_2} \sigma_z^{p_2} \dots \otimes \sigma_x^{q_n} \sigma_z^{p_n} \quad (12)$$

As in the continuous case, the DWF is translationally covariant. Let  $W$  and  $W'$  be the DWF associated with the states  $\rho$  and  $\rho'$ . If  $\rho$  and  $\rho'$  are related by the unitary translation operator  $T_\beta$  by,

$$\rho' = \hat{T}_\beta \rho \hat{T}_\beta^\dagger \quad (13)$$

then :

$$W'_\alpha = W_{\alpha+\beta} \quad (14)$$

Thus,  $W'$  is obtained by shifting each element of  $W$  by an amount  $\beta$

### III. SPIN FLIPPED DWF OF A MULTIQUBIT SYSTEM

To derive a relationship between the DWF  $W$  and its spin-flipped counterpart  $\tilde{W}$ , we begin by observing that the spin flip operation on the density matrix is a two step process: the first step involves the complex conjugation of  $\rho$  in the computational basis and the next one entails the application of the translational operators  $\sigma_y^{\otimes n}$  to it. Consequently, the computation of the spin flipped DWF  $\tilde{W}$  can be carried out in two steps. We denote the DWF of  $\rho^*$  by  $W^{(*)}$  (not to be confused with complex conjugation of  $W$ , which in any case is real valued). In the first step, we derive an expression for  $W^{(*)}$  in terms of  $W$ . A shift associated with the translation  $\sigma_y^{\otimes n}$  of  $\rho^*$  is then effected by subjecting  $W^{(*)}$  to a corresponding shift, to obtain  $\tilde{W}$ .

### A. Derivation of $W^{(*)}$ for the multiqubit state

If the system is represented by a state vector or a density matrix, then complex conjugation is straight forward. The procedure is however not obvious when the system is represented by the DWF. To elucidate this, we exploit the relationship between  $W$  and  $\rho$  and write  $W_\alpha^{(*)}$  as:

$$W_\alpha^{(*)} = \frac{1}{N} \text{Tr}(\rho^* A_\alpha) \quad (15)$$

since  $A_\alpha$  are self adjoint operators. Now taking the complex conjugate of Eq (5) we have:

$$\rho^* = \sum_{\beta} W_{\beta} A_{\beta}^* \quad (16)$$

substituting Eq.(15) in Eq. (16), thus:

$$W_\alpha^{(*)} = \frac{1}{N} \text{Tr} \left[ \left( \sum_{\beta} W_{\beta} A_{\beta}^* \right) A_{\alpha} \right] \quad (17)$$

taking the trace operation inside the summation:

$$W_\alpha^{(*)} = \frac{1}{N} \sum_{\beta} W_{\beta} \text{Tr} (A_{\alpha} A_{\beta}^*) \quad (18)$$

writing the Wigner elements as a column vector and the Wigner elements in terms of the field labels:

$$\begin{pmatrix} W_{0,0}^{(*)} \\ \vdots \\ W_{1,0}^{(*)} \\ \vdots \\ W_{\bar{\omega},\bar{\omega}}^{(*)} \end{pmatrix} = \frac{1}{N} \begin{pmatrix} \text{Tr}(A_{0,0} A_{0,0}^*) & \text{Tr}(A_{0,0} A_{0,1}^*) & \cdots & \cdots & \text{Tr}(A_{0,0} A_{\bar{\omega},\bar{\omega}}^*) \\ \text{Tr}(A_{0,1} A_{0,0}^*) & \text{Tr}(A_{0,1} A_{0,1}^*) & \cdots & \cdots & \text{Tr}(A_{0,1} A_{\bar{\omega},\bar{\omega}}^*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \text{Tr}(A_{\bar{\omega},\bar{\omega}} A_{0,0}^*) & \text{Tr}(A_{\bar{\omega},\bar{\omega}} A_{0,1}^*) & \cdots & \cdots & \text{Tr}(A_{\bar{\omega},\bar{\omega}} A_{\bar{\omega},\bar{\omega}}^*) \end{pmatrix} \begin{pmatrix} W_{0,0} \\ \vdots \\ W_{1,0} \\ \vdots \\ W_{\bar{\omega},\bar{\omega}} \end{pmatrix} \quad (19)$$

This equation can be written in a compact form:

$$W^{(*)} = S W \quad (20)$$

where  $S$  is a  $N^2 \times N^2$  matrix. Thus, Eq. 20 helps us to implement the complex conjugation operation for the DWFs. In the next section we show that the matrix  $S$  is a Hadamard matrix and that it is independent of the quantum net.

### B. Proof that $S$ is a Hadamard Matrix and that it is independent of the quantum net

While the DWF depends on the specific choice of the quantum net, we show that the transformation matrix  $S$  is itself independent of this choice. Thus, we shall show that a single Hadamard matrix  $S$  is sufficient for transforming  $W$ , obtained using any quantum net, to the corresponding  $W^{(*)}$ . It may easily be verified that the transformation matrix  $S$  is made of  $N \times N$  blocks, of size  $N \times N$ . We shall label the blocks of  $S$  as  $S_{ij}$  where the suffixes are the block indices. Note however that  $S(k,l)$  denote the matrix elements. For example, the first block  $S_{11}$  is a  $N \times N$  matrix formed by varying the elements  $\alpha$  and  $\beta$  in  $\text{Tr}(A_{\alpha} A_{\beta}^*)$  over the points in the first line of the first striation. We now start by writing the explicit equation for  $\text{Tr}(A_{\alpha} A_{\beta}^*)$ . Using Eq(9),

$$\begin{aligned} \text{Tr}(A_{\alpha} A_{\beta}^*) &= \text{Tr} \left[ \left( \sum_{\lambda \ni \alpha} Q(\lambda) - I \right) \left( \sum_{\lambda' \ni \beta} Q^*(\lambda') - I \right) \right] \\ &= \text{Tr} \{ [Q(\lambda_1^{x_1}) + \cdots + Q(\lambda_{N+1}^{x_{N+1}})] [Q^*(\lambda_1^{y_1}) + \cdots + Q^*(\lambda_{N+1}^{y_{N+1}})] \} - N - 2 \\ &= T_{\alpha\beta} - N - 2 \end{aligned} \quad (21)$$

where the term  $T_{\alpha\beta}$  contains  $(N+1)^2$  terms of the form  $Tr[Q(\lambda_i^{x_i})Q^*(\lambda_j^{y_j})]$ . The  $Q(\lambda_i^{x_i})$ s are rank one projectors associated with the  $x_i^{th}$  line in the  $i^{th}$  striation. In the  $N+1$  MUBS, the first basis sets can always be taken to have real elements and hence complex conjugation does not alter them. Therefore, for these two cases,  $Tr[Q(\lambda_i^{x_i})Q^*(\lambda_i^{x_i})] = 1$  (where  $i \in [1, 2]$ ) as  $Q^*(\lambda_i^{x_i}) = Q(\lambda_i^{x_i})$ . The other  $N-1$  bases have complex entries and are closed under complex conjugation i.e., complex conjugation takes each basis vector to another one orthogonal to it. Therefore,  $Tr[Q(\lambda_i^{x_i})Q^*(\lambda_i^{x_i})] = 0$  for  $i \in [3, 4, \dots, N+1]$ . If however,  $Q(\lambda_i^{x_i})$  and  $Q^*(\lambda_j^{y_j})$  belong to different striations i.e., different bases sets, then,  $Tr[Q(\lambda_i^{x_i})Q^*(\lambda_j^{y_j})] = \frac{1}{N}$  ( $i \neq j$ ). It is important to note that for all these cases,  $Tr[Q(\lambda_i^{x_i})Q^*(\lambda_j^{y_j})]$  is independent of the quantum net. It essentially arises from the fact that DWF is translationally covariant.

Consider the case  $\alpha = \beta$  in  $Tr(A_\alpha A_\beta^*)$ , which are the set of all diagonal elements of the matrix  $S$ . From Eq (21)  $Tr(A_\alpha A_\beta^*) = T_{\alpha\beta} - N - 2$ , where the term  $T_{\alpha\beta}$  has  $(N+1)^2$  trace terms. There are  $N(N+1)$  trace terms  $Tr[Q(\lambda_i^{x_i})Q^*(\lambda_j^{y_j})] = \frac{1}{N}$  for which  $i \neq j$ , that is  $Q(\lambda_i^{x_i})$  and  $Q^*(\lambda_j^{y_j})$  are from different striations, so their value become  $N(N+1)\frac{1}{N} = N+1$ . In the remaining  $N+1$  terms, the value of  $Tr[Q(\lambda_i^{x_i})Q^*(\lambda_i^{x_i})]$  is 1 for the first two striations and 0 for the other  $N-1$  striations. Therefore  $Tr(A_\alpha A_\alpha^*) = 1$  for all  $\alpha$ , so that all the diagonal entries of the  $S$  matrix are all equal to 1.

Next, consider the case in which  $\alpha$  and  $\beta$  belong to the same line in the first striation. This spans all the diagonal blocks of the  $S$  matrix. The variable  $T_{\alpha\beta}$  in  $Tr(A_\alpha A_\beta^*)$  is  $N(N+1)$  times  $\frac{1}{N}$ , since  $Tr[Q(\lambda_i^{x_i})Q^*(\lambda_j^{y_j})] = \frac{1}{N}$ , for which,  $i \neq j$ . Since  $\alpha$  and  $\beta$  are from the same striation, for each value of  $\alpha$  and  $\beta$ , there are two trace terms which contribute 1 and other terms are zero. So in this case  $Tr(A_\alpha A_\beta^*) = 1$ . Therefore, all the diagonal blocks of  $S$  are  $N \times N$  matrices with unit entries.

As the last case, consider  $\alpha$  and  $\beta$  belonging to the different lines of the first striation. This condition spans all the off-diagonal blocks of the  $S$  matrix. Here too, the calculation of  $T_{\alpha\beta}$  gives  $N(N+1)$  times  $\frac{1}{N}$  for the  $i \neq j$  case. Since  $\alpha$  and  $\beta$  are from two distinct lines of the first striation  $Tr[Q(\lambda_1^{x_1})Q^*(\lambda_1^{y_1})] = 0, \forall \alpha, \beta$ . For the given  $\alpha$ , if  $\beta$  runs over all the points in the given striation, it gives a particular row in that block. In that row  $\frac{N}{2}$  entries are +1s and  $\frac{N}{2}$  are -1s.

Thus, in  $S$ , all the diagonal blocks are  $N \times N$  matrices have entries which are +1 and all the off-diagonal blocks are the  $N \times N$  matrix with each row containing an equal number of +1s and -1s. Further, the rows and columns of  $S$  are orthogonal to each other. We thus see that  $S$  is a Hadamard matrix. We note that these observations are not specific to any quantum net, implying that  $S$  is independent of the same.

### C. The spin flipped Wigner function $\tilde{W}$

The next step now is to obtain the spin flipped DWF  $\tilde{W}$  from  $W^{(*)}$ . In terms of density matrices, the spin flip operation is defined as  $\tilde{\rho} = \sigma_y^{\otimes n} \rho^* \sigma_y^{\otimes n}$ . To find the  $\tilde{W}$ , the Pauli operators acting on the individual qubits are effectively translation operators  $T_\beta$  acting on  $W^{(*)}$ , that shift each element of  $W^{(*)}$  in the phase space by an amount  $\beta$ . We can realize this transformation by a  $N^2 \times N^2$  matrix  $T$  acting on  $W^{(*)}$  (arranged as a column vector) to obtain  $\tilde{W}$ :

$$\tilde{W} = TW^{(*)} \quad (22)$$

$T$  depends on the basis choice of the underlying Galois field. Using the Eq (20) we can write  $\tilde{W}$  directly in terms of  $W$  as:

$$\tilde{W} = TSW \quad (23)$$

$T$  acting on  $S$  merely interchanges the rows, and so the resultant matrix is again a Hadamard matrix  $H = TS$ . Thus, the spin flipped state takes the form:

$$\tilde{W} = HW \quad (24)$$

This completes the derivation of  $\tilde{W}$  from  $W$  and the proof that the Hadamard matrix  $H$  is quantum net independent.

## IV. ILLUSTRATION OF THE SPIN FLIP OPERATION FOR ONE AND TWO QUBIT DWFS

### A. Illustration for a one qubit system

In order to clarify the procedure for computing  $\tilde{W}$ , let us consider the spin flip operation on a one qubit system. For this case, the axes of the discrete phase space are labelled by the Galois field elements  $\mathbb{F}_2 = \{0, 1\}$ . In the  $2 \times 2$

phase space, there are 3 striations and each striation contains two lines, having totally 6 lines. The MUBs associated with the 3 striations are,

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad B_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

where the matrices are the basis sets and the columns of  $B_i$  are the mutually orthogonal basis vectors. The first two basis sets  $B_1$  and  $B_2$  are not altered by complex conjugation, but for the third basis set  $B_3$ , complex conjugation interchanges columns 1 and 2. From Eq (20) we have  $W^{(*)} = SW$ .  $S$  can be therefore be written as,

$$S = \frac{1}{2} \begin{pmatrix} Tr(A_{0,0}A_{0,0}^*) & Tr(A_{0,0}A_{0,1}^*) & Tr(A_{0,0}A_{1,0}^*) & Tr(A_{0,0}A_{1,1}^*) \\ Tr(A_{0,1}A_{0,0}^*) & Tr(A_{0,1}A_{0,1}^*) & Tr(A_{0,1}A_{1,0}^*) & Tr(A_{0,1}A_{1,1}^*) \\ Tr(A_{1,0}A_{0,0}^*) & Tr(A_{1,0}A_{0,1}^*) & Tr(A_{1,0}A_{1,0}^*) & Tr(A_{1,0}A_{1,1}^*) \\ Tr(A_{1,1}A_{0,0}^*) & Tr(A_{1,1}A_{0,1}^*) & Tr(A_{1,1}A_{1,0}^*) & Tr(A_{1,1}A_{1,1}^*) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \quad (25)$$

where  $S$  is written as a block matrix with  $2 \times 2$  blocks, each block being a  $2 \times 2$  matrix. By examining Eq (21), clearly,  $Tr(A_\alpha A_\beta^*) = T_{\alpha\beta} - 4$  and,

- For points  $\alpha = \beta$ , i.e for the diagonal entries of  $S_{11}$  and  $S_{22}$ ,  $Tr[Q(\lambda_i^{x_i})Q^*(\lambda_i^{x_i})] = 1$  for  $i = 1, 2$  and  $Tr[Q(\lambda_i^{x_i})Q^*(\lambda_i^{y_i})] = 0$  for  $i = 3$ . Therefore,  $T_{\alpha\alpha} = [2(2+1)\frac{1}{2}] + 1 + 1 = 5$ . So,  $Tr(A_\alpha A_\alpha^*) = 1$ . Therefore, for all the diagonal entries  $Tr(A_\alpha A_\alpha^*) = 1$ .
- If  $\alpha$  and  $\beta$  belong to the same line of the first striation then,  $Tr[Q(\lambda_1^{x_1})Q^*(\lambda_1^{x_1})] = 1$ , whenever  $Tr[Q(\lambda_2^{x_2})Q^*(\lambda_2^{x_2})] = 1$  then  $Tr[Q(\lambda_3^{x_3})Q^*(\lambda_3^{x_3})] = 0$  and vice versa. Therefore,  $T_{\alpha\beta} = [2(2+1)\frac{1}{2}] + 1 + 1 = 5$ . So  $Tr(A_\alpha A_\beta^*) = 1$ . This argument holds for all the diagonal blocks. Thus, the diagonal blocks  $S_{11}$  and  $S_{22}$  are of the form:  $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ .
- For the off diagonal blocks,  $\alpha$  and  $\beta$  belong to different lines of the first striation and  $Tr[Q(\lambda_1^{x_1})Q^*(\lambda_1^{y_1})] = 0$  for all  $\alpha$  and  $\beta$ . If points  $\alpha$  and  $\beta$  belong to the same line of the third striation, then,  $Tr[Q(\lambda_3^{x_3})Q^*(\lambda_3^{x_3})] = 0$ . Therefore,  $T_{\alpha\beta} = [2(2+1)\frac{1}{2}] = 3$ . So  $Tr(A_\alpha A_\beta^*) = -1$ , otherwise  $Tr(A_\alpha A_\beta^*) = 1$ . For example  $\alpha = (0, 0)$  and  $\beta = (1, 1)$  belong to different lines of the first striation and same line of the third striation, so  $Tr(A_\alpha A_\beta^*) = -1$ .

The transformation matrix  $S$  thus takes the form,

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \quad (26)$$

In the construction of this matrix, we have not used any particular quantum net, hence  $S$  is independent of the same. The matrix  $S$  has the following properties:

1. The elements of the  $S$  are either  $\pm \frac{1}{N}$ , where  $N = 2^n$ .
2. Two different rows(columns) are orthogonal to each other.
3. Determinant of the  $S$  is  $-1$ . For a general  $n$  qubit system, the determinant of  $S$  is equal to  $(-1)^n$ .
4. It is a self-inverse.

therefore, the DWF of the complex conjugated state can be written as,

$$\begin{pmatrix} W_{00}^{(*)} \\ W_{01}^{(*)} \\ W_{10}^{(*)} \\ W_{11}^{(*)} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} W_{00} \\ W_{01} \\ W_{10} \\ W_{11} \end{pmatrix} \quad (27)$$

The next task is to perform the unitary translational operation on  $W^{(*)}$ , to complete the spin flip operation. In the equation  $\tilde{\rho} = \sigma_y \rho * \sigma_y^\dagger$ , the Pauli's operator  $\sigma_y$  is essentially a translational operator  $T_\beta = \sigma_y$ , which translates every point in the DWF  $W^{(*)}$  by an amount  $\beta$ . The spin flipped state is hence given by  $\tilde{W}_\alpha = W_{\alpha+\beta}^{(*)}$ . From Eq (12), it is seen that the general translational operator on phase space can be written as  $T_{(q,p)} = \sigma_x^q \sigma_z^p$ . From this equation,  $\sigma_y$



may be expressed as  $T_\beta = \sigma_y = -i(\sigma_x^1 \sigma_z^1) = T_{(1,1)}$ , where  $-i$  is the phase factor. Therefore, the spin flip operation is carried out by translating every element of the phase space by an amount  $\beta = (1, 1)$ . This translation can also be represented by a matrix :

$$\begin{pmatrix} \tilde{W}_{00} \\ \tilde{W}_{01} \\ \tilde{W}_{10} \\ \tilde{W}_{11} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} W_{00}^{(*)} \\ W_{01}^{(*)} \\ W_{10}^{(*)} \\ W_{11}^{(*)} \end{pmatrix} \quad (28)$$

Therefore  $\tilde{W}$  may be written as  $\tilde{W} = TW$ . We know that  $W^{(*)} = SW$ , hence using Eq (27), and Eq (28), we may write,

$$\begin{pmatrix} \tilde{W}_{00} \\ \tilde{W}_{01} \\ \tilde{W}_{10} \\ \tilde{W}_{11} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} W_{00} \\ W_{01} \\ W_{10} \\ W_{11} \end{pmatrix} \quad (29)$$

or

$$\tilde{W} = HW \quad (30)$$

The product  $H = TS$ , is also a Hadamard matrix as explained earlier.

### B. Spin flipped DWF of a two qubit system

To appreciate the general results obtained for the multiqubit state, it would be helpful to additionally consider the two qubit case in some detail. The points in the axis of two qubit discrete phase space are labelled by the finite field  $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$ . The MUBS associated with this discrete phase space is given by:

$$\begin{aligned} B_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} B_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} B_3 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -i & i & -i & i \\ i & i & -i & -i \\ 1 & -1 & -1 & 1 \end{pmatrix} \\ B_4 &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ i & i & -i & -i \\ -i & i & i & -i \end{pmatrix} B_5 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -i & i & -i & i \\ 1 & 1 & -1 & -1 \\ i & -i & -i & i \end{pmatrix} \end{aligned}$$

As in the single qubit case, each matrix is a basis set and the columns are the basis vectors. Since the first two basis sets do not have any complex entries, complex conjugation does not alter them. But the last three bases have complex entries and complex conjugation takes a vector into some other vector in the same basis. For two qubit systems, the discrete phase space is a  $4 \times 4$  array of points having 16 entries, Writing  $W$  as a  $16 \times 1$  column vector, Using Eq (20) we have:

$$W^{(*)} = SW$$

where  $S$  is a  $16 \times 16$  matrix. This  $16 \times 16$  matrix can be considered as constituted of  $4 \times 4$  blocks where each block is again a  $4 \times 4$  matrix and denote the blocks by  $S_{ij}$ . The  $S_{11}$  block is constructed by running through the points of the first line in the first striation i.e.,  $(0, 0)$ ,  $(0, 1)$ ,  $(0, \omega)$  and  $(0, \bar{\omega})$ . So,  $S_{11}$  is given by,

$$S_{11} = \begin{pmatrix} Tr(A_{0,0}A_{0,0}^*) & Tr(A_{0,0}A_{0,1}^*) & Tr(A_{0,0}A_{0,\omega}^*) & Tr(A_{0,0}A_{0,\bar{\omega}}^*) \\ Tr(A_{0,1}A_{0,0}^*) & Tr(A_{0,1}A_{0,1}^*) & Tr(A_{0,1}A_{0,\omega}^*) & Tr(A_{0,1}A_{0,\bar{\omega}}^*) \\ Tr(A_{0,\omega}A_{0,0}^*) & Tr(A_{0,\omega}A_{0,1}^*) & Tr(A_{0,\omega}A_{0,\omega}^*) & Tr(A_{0,\omega}A_{0,\bar{\omega}}^*) \\ Tr(A_{0,\bar{\omega}}A_{0,0}^*) & Tr(A_{0,\bar{\omega}}A_{0,1}^*) & Tr(A_{0,\bar{\omega}}A_{0,\omega}^*) & Tr(A_{0,\bar{\omega}}A_{0,\bar{\omega}}^*) \end{pmatrix} \quad (31)$$

In this block matrix, the value of  $Tr(A_\alpha A_\beta^*) = T_{\alpha\beta} - N - 2$ , where,

$$T_{\alpha\beta} = Tr[Q(\lambda_1^{x_1})Q^*(\lambda_1^{x_1})] + Tr[Q(\lambda_1^{x_1})Q^*(\lambda_2^{x_2})] + \dots + Tr[Q(\lambda_{N+1}^{x_{N+1}})Q^*(\lambda_{N+1}^{x_{N+1}})] \quad (32)$$



In  $T_{\alpha\beta}$ , there are  $(4+1)^2$  trace terms, wherein  $Q(\lambda_i^{x_i})$  and  $Q^*(\lambda_j^{x_j})$  are from different striations, for which  $Tr[Q(\lambda_1^{x_1})Q^*(\lambda_2^{x_2})]$  is  $\frac{1}{4}$ . In general there are  $4(4+1)$  such trace terms in  $T_{\alpha\beta}$ . In the remaining  $(4+1)$  terms,  $Tr[Q(\lambda_i^{x_i})Q^*(\lambda_i^{y_i})]$  is either 0 or 1 depending on the points  $\alpha$  and  $\beta$ . For example,

- if  $\alpha = (0,0)$  and  $\beta = (0,0)$ ,  $Tr[Q(\lambda_i^{x_i})Q^*(\lambda_i^{y_i})] = 1$  for  $i = 1, 2$  and  $Tr[Q(\lambda_i^{x_i})Q^*(\lambda_i^{y_i})] = 0$  for  $i = 3, 4, 5$ . Therefore  $T_{\alpha\beta} = [4(4+1)\frac{1}{4}] + 1 + 1 = 7$ . So  $Tr(A_{0,0}A_{0,0}^*) = 1$ . This argument holds for all the diagonal entries.
- if  $\alpha = (0,0)$  and  $\beta = (0,\omega)$ ,  $Tr[Q(\lambda_i^{x_i})Q^*(\lambda_i^{y_i})] = 1$  for  $i = 1, 3$  and  $Tr[Q(\lambda_i^{x_i})Q^*(\lambda_i^{y_i})] = 0$  for  $i = 2, 4, 5$ . Therefore  $T_{\alpha\beta} = [4(4+1)\frac{1}{4}] + 1 + 1 = 7$ . So  $Tr(A_{0,0}A_{0,\omega}^*) = 1$ . Similar argument holds for all the off-diagonal entries of the diagonal blocks.

Therefore, all the diagonal blocks are  $4 \times 4$  matrix whose entries are 1. Next, consider one block from the off-diagonal blocks-  $S_{12}$  is one of them:

$$S_{12} = \begin{pmatrix} Tr(A_{0,0}A_{1,0}^*) & Tr(A_{0,0}A_{1,1}^*) & Tr(A_{0,0}A_{1,\omega}^*) & Tr(A_{0,0}A_{1,\bar{\omega}}^*) \\ Tr(A_{0,1}A_{1,0}^*) & Tr(A_{0,1}A_{1,1}^*) & Tr(A_{0,1}A_{1,\omega}^*) & Tr(A_{0,1}A_{1,\bar{\omega}}^*) \\ Tr(A_{0,\omega}A_{1,0}^*) & Tr(A_{0,\omega}A_{1,1}^*) & Tr(A_{0,\omega}A_{1,\omega}^*) & Tr(A_{0,\omega}A_{1,\bar{\omega}}^*) \\ Tr(A_{0,\bar{\omega}}A_{1,0}^*) & Tr(A_{0,\bar{\omega}}A_{1,1}^*) & Tr(A_{0,\bar{\omega}}A_{1,\omega}^*) & Tr(A_{0,\bar{\omega}}A_{1,\bar{\omega}}^*) \end{pmatrix} \quad (33)$$

Here too, the crucial step is in calculating  $T_{\alpha\beta}$ . In  $T_{\alpha\beta}$ , there are  $4(4+1)$  terms with value  $\frac{1}{4}$ . Since this block is formed by the first and second lines in the first striation, the trace term  $Tr[Q(\lambda_1^{x_1})Q^*(\lambda_1^{y_1})] = 0$ . For a given  $\alpha$ , with the point  $\beta$  running over the points in the second line in the first striation, 4 times the trace terms  $Tr[Q(\lambda_i^{x_i})Q^*(\lambda_i^{y_i})]$  becomes 1 for  $i = 2, \dots, 5$ . For  $\alpha = (0,0)$  and  $\beta = (1,0)$ ,  $Tr[Q(\lambda_i^{x_i})Q^*(\lambda_i^{y_i})] = 1$  for  $i = 2, 4$  therefore,  $Tr(A_{0,0}A_{1,0}^*) = 1 + 1 + (N+1) - N - 2 = 1$ . Similarly, for  $\alpha = (0,0)$  and  $\beta = (1,\omega)$ ,  $Tr[Q(\lambda_i^{x_i})Q^*(\lambda_i^{y_i})] = 1$  for  $i = 3, 5$  therefore  $Tr(A_{0,0}A_{1,\omega}^*) = 1 + 1 + (N+1) - N - 2 = 1$ . The remaining terms in that row are  $-1$ , that is  $Tr(A_{0,0}A_{1,1}^*) = -1 = Tr(A_{0,0}A_{1,\bar{\omega}}^*)$ . Same arguments holds for the other rows of this block, that is the term  $Tr(A_{\alpha}A_{\beta}^*)$  becomes two times  $+1$  and the other two times  $-1$ . So the block matrix takes the form,

$$S_{12} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \quad (34)$$

Since  $Tr(A^*B) = Tr(AB^*)$ , if  $A$  and  $B$  are Hermitian matrices, the block matrix  $S_{ji} = S_{ij}^T$ , where  $T$  stands for transposition. The columns of the off-diagonal blocks has equal number of  $+1$  and  $-1$ . So, the general transformation matrix has entries which are either  $+1$  or  $-1$ . And rows(columns) of  $S$  are orthogonal to each other. Thus, the  $S$  is a Hadamard matrix of dimension  $16 \times 16$ . This transforms a DWF ( $16 \times 1$  column vector) to a complex conjugate DWF by,

$$W^{(*)} = SW$$

The second step, involves the calculation of  $\tilde{W}$  in terms of  $W^{(*)}$ , where  $\tilde{W}$  is the DWF of  $\tilde{\rho}$ . We may recall, that the spin flipped density matrix  $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$  as given by Eq (2). The effect of the translation operator  $T_{\beta} = -\sigma_y \otimes \sigma_y$  on  $\rho^*$  is to cause a shift in  $W_{\alpha}^{(*)}$  by an amount  $\beta$ , which depends on the basis choice for the field. From Eq (12) the translational operator  $T_{(q,p)}$  in discrete phase space can then be written as,

$$T_{(q,p)} = \sigma_x^{q_1} \sigma_z^{p_1} \otimes \sigma_x^{q_2} \sigma_z^{p_2} \quad (35)$$

Since  $\sigma_y = i\sigma_z\sigma_x$ , we can rewrite the translational operator  $T_{\beta} = -\sigma_y \otimes \sigma_y$  by:

$$T_{(q,p)} = \sigma_x\sigma_z \otimes \sigma_x\sigma_z \quad (36)$$

From Eq (35) & Eq (36) it is clear that  $q_1 = p_1 = q_2 = p_2 = 1$ . If we choose  $(\omega, 1)$  as the basis of the horizontal axis, then the basis elements for the vertical axis also turns out to be  $(\omega, 1)$ . Therefore  $q$  and  $p$  can be expressed in terms of the basis as:

$$q = q_1.e_1 + q_2.e_2 = 1.\omega + 1.1$$

Table I. The DWF of  $W^*$  subjected to rigid translation effected by  $\sigma_y \otimes \sigma_y$ , results in the shifting of the elements of  $W^*$  by  $\beta = (\bar{\omega}, \bar{\omega})$  to yield the corresponding element of  $\tilde{W}$

$W^*$					TS	$\tilde{W}$				
- -	$W_{0,\bar{\omega}}^*$	$W_{1,\bar{\omega}}^*$	$W_{\omega,\bar{\omega}}^*$	$W_{\bar{\omega},\bar{\omega}}^*$	$\xrightarrow{\sigma_y \otimes \sigma_y}$	- -	$W_{0,\bar{\omega}}^*$	$W_{\omega,0}^*$	$W_{1,0}^*$	$W_{0,0}^*$
- +	$W_{0,\omega}^*$	$W_{1,\omega}^*$	$W_{\omega,\omega}^*$	$W_{\bar{\omega},\omega}^*$		- +	$W_{\bar{\omega},1}^*$	$W_{\omega,1}^*$	$W_{1,1}^*$	$W_{0,1}^*$
+ -	$W_{0,1}^*$	$W_{1,1}^*$	$W_{\omega,1}^*$	$W_{\bar{\omega},1}^*$		+ -	$W_{\omega,\omega}^*$	$W_{\bar{\omega},\omega}^*$	$W_{1,\omega}^*$	$W_{0,\omega}^*$
+ +	$W_{0,0}^*$	$W_{1,0}^*$	$W_{\omega,0}^*$	$W_{\bar{\omega},0}^*$		+ +	$W_{\bar{\omega},\bar{\omega}}^*$	$W_{\omega,\bar{\omega}}^*$	$W_{1,\bar{\omega}}^*$	$W_{0,\bar{\omega}}^*$
	HH	HV	VH	VV			HH	HV	VH	VV

$$q = \omega + 1 = \bar{\omega} \quad (37)$$

and

$$p = p_1 \cdot f_1 + p_2 \cdot f_2$$

$$p = \omega + 1 = \bar{\omega} \quad (38)$$

So, for this particular choice of bases  $\beta = (\bar{\omega}, \bar{\omega})$  and the translational operator  $T_\beta = -\sigma_y \otimes \sigma_y = T_{(\bar{\omega}, \bar{\omega})}$ . Therefore,

$$\tilde{W}_\alpha = W_{\alpha+\beta}^{(*)} \quad (39)$$

where  $\beta = (\bar{\omega}, \bar{\omega})$ .

Therefore, from Table 1, we see that,  $\tilde{W}$  can be calculated from  $W^*$  just by translating the elements of  $W^*$  by  $\beta = (\bar{\omega}, \bar{\omega})$ . If the Wigner elements are arranged as a column vector then the this translation is carried out by,

$$\tilde{W} = TW^{(*)} \quad (40)$$

But the DWF of the complex conjugate state is already a Hadamard transformation of the original DWF, that is  $W^{(*)} = SW$ . So this Eq can be written as,

$$\tilde{W} = TSW \quad (41)$$

$$\tilde{W} = HW \quad (42)$$

It is fairly straight forward to show that the product of matrices  $T$  and  $S$  is again a Hadamard matrix. We have thus illustrated the underlying method using the one qubit and two qubit states as examples. From these illustrations, it would be clear how the Hadamard matrix  $H$  may be pre-computed for any arbitrary multiqubit state.

## V. CONCLUSIONS

The experimental measurement of continuous Wigner function has been extensively reported in the quantum optics literature. In these studies, quantum interference effects are beautifully brought out and a link between non-classicality and negativity of the Wigner function is transparent. However, unlike the continuous case, the discrete Wigner function has not been as thoroughly investigated and barring some examples, its utility is not all together clear. One of the important reasons for this stems from the fact that the DWF is not unique and different quantum nets give rise to different DWFs. Since the DWF depends on the choice of the quantum net, the non-classicality or otherwise of the reconstructed state is not obvious. Evidently, a clear interpretation of the consequences of spin flip would require the derivation of quantities that are independent of the choice of the underlying quantum net. In the present work we have shown that the DWF and the spin-flipped DWF of the multiqubit states are related through a linear transform involving a Hadamard matrix. We have further shown that this matrix is independent of the choice of the quantum net used in the reconstruction of the DWF. We have illustrated this for the one and two qubit discrete

Wigner functions. Experimentally, several protocols are available for the tomographic reconstruction of the DWF but there are no entanglement measures defined purely in terms of discrete Wigner elements. One way of defining entanglement measures for DWFs is use those defined for  $\rho$  and find equivalent expressions in terms of the DWF elements. With the present results, we can readily compute bipartite concurrence in terms of the DWF using the definition given in equation 1. With a bit of algebra, it can be shown that the matrix  $R = \rho\tilde{\rho}$  may be written as  $R = \rho\tilde{\rho} = \frac{1}{4} \sum_{\alpha} \sum_{\gamma} W_{\alpha} \tilde{W}_{\beta} \hat{A}_{\gamma} \hat{A}_{\delta}$ . Likewise, one could also rewrite the expressions of the other tangles in terms of the DWF. Alternately one may attempt to define altogether SLOCC invariant measures starting from the DWF. We thus hope that the present work may open a way to understanding entanglement directly in terms of the DWFs.

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